QWORLD

INTRODUCTION TO CLASSICAL SYSTEMS

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An engineer, a physicist and a mathematician…

What is Your Favourite Super Power?

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MANIPULATE PROBABILITY!!!

Deterministic Classical Information

A physical device X has some finite, non-empty set of states.

$\{H, T\}$ $\quad \Sigma = \{0, 1\}$ $\quad \{a, b, c, d\}$

How does the state of the system change?

$f : \{0,1\} \rightarrow \{0,1\}$

x	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

$\Sigma^2 = \{00, 01, 10, 11\}$ **What about multiple such devices?**

$|\Sigma^n| = 2^n$

$\underbrace{\underbrace{2}\times\underbrace{2}\times\dots\times\underbrace{2}}_{n}$

2^n possibilities

x_1	x_2	$\dots$	x_n	f
0	0	$\dots$	0	0
0	0	$\dots$	1	1
		$\vdots$		$\vdots$
1	1	$\dots$	1	0

$f : \{0,1\}^n \longrightarrow \{0,1\}$

$N = 2^n$

$2^N = 2^{2^n}$ # of functions

What if we have Incomplete Information?

What if we have Incomplete Information?

Probability

$\hat{v}$

$$\mathrm{Probability}(\text{State of } X = 0) = p$$
$$\mathrm{Probability}(\text{State of } X = 1) = 1-p$$

$$\hat{v} = \begin{array}{c} \begin{pmatrix} p \\ 1-p \end{pmatrix} \end{array} \begin{array}{l} \to v_0 \\ \to v_1 \end{array}$$

Note: When we look at X we do not see $\hat{v}, p$ but rather some state $\sigma \in \Sigma$ with probability v_σ

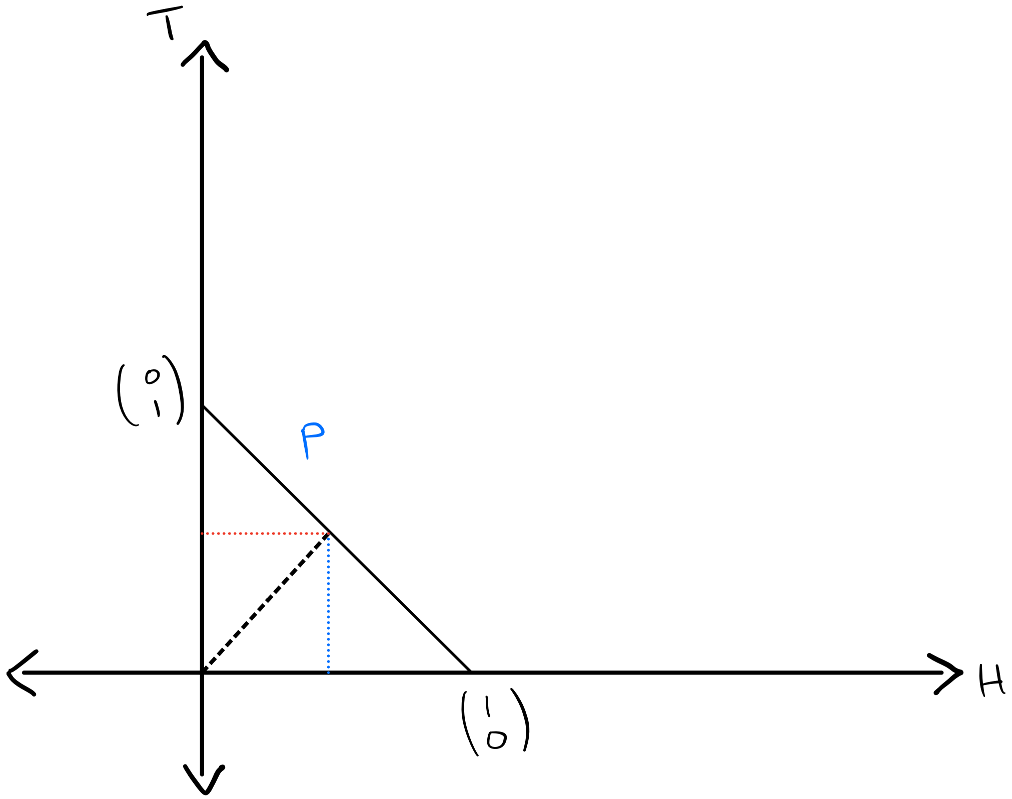
What if we have Incomplete Information?

$$\hat{v} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$$= \underbrace{p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{convex combination}}$$

$p + (1-p) = 1$

$p \geq 0$



What if we have Incomplete Information?

$p(H|T)$

$p(H|H)$

$$\begin{array}{cc} & \begin{array}{cc} H & \quad T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & d \end{pmatrix} \end{array} \begin{pmatrix} p \\ 1-p \end{pmatrix} \begin{array}{c} H \\ T \end{array} = \underbrace{\begin{pmatrix} ap + b(1-p) \\ cp + d(1-p) \end{pmatrix}}_{\text{This should be a probability vetor.}}$$

$p(T|H)$

$p(T|T)$

A classical transformation is <u>stochastic</u>

1) all entries ≥ 0
2) Sum of columns $= 1$

What if we have Incomplete Information?

Deterministic Ops as Matrices

$$A_0 = \begin{array}{c|cc} & H & T \\ \hline H & 1 & 1 \\ T & 0 & 0 \end{array} \;\to\; \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

always 'H'   identity   NOT   always 'T'

e.g., $A_3 \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix}$

How does the State of the System change?

Probabilist op.

$$B = \sum_{i=0}^{3} p_i A_i \quad , \; p_i \geq 0, \; \sum p_i = 1$$

e.g. $F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

How does the State of the System change?

Qs. We flip a coin $\begin{array}{c c} T & \\ H & \end{array}\!\begin{pmatrix} p \\ q \end{pmatrix}$.
$\begin{cases} \text{if we get a head, we flip again.} \\ \text{if we get a tail, we turn the coin over, i.e., we make it a head.} \end{cases}$

$$\begin{array}{c c} & \begin{array}{cc} T & H \end{array} \\ \begin{array}{c} T \\ H \end{array} & \begin{pmatrix} & p \\ 1 & q \end{pmatrix} \end{array} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} pq \\ p+q^2 \end{pmatrix}$$

Multiple Devices with Incomplete Information

Coin X_1 $\begin{array}{cc} \begin{pmatrix} p \\ q \end{pmatrix} & \begin{array}{c} H \\ T \end{array} \end{array}$

Coin X_2 $\begin{array}{cc} \begin{pmatrix} r \\ s \end{pmatrix} & \begin{array}{c} H \\ T \end{array} \end{array}$

Tensor Product

$$\begin{pmatrix} p \\ q \end{pmatrix} \otimes \begin{pmatrix} r \\ s \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} pr \\ ps \\ qr \\ qs \end{pmatrix} & \begin{array}{c} HH \\ HT \\ TH \\ TT \end{array} \end{array}$$

Note Not all 4-dim probability vectors can be written in the above form.

$$p(x_1 x_2) \neq p(X_1)\,p(X_2)$$

e.g. $$\frac{1}{2} \begin{array}{cc} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{array}{c} HH \\ \\ \\ TT \end{array} \end{array} \neq \begin{pmatrix} p \\ q \end{pmatrix} \otimes \begin{pmatrix} r \\ s \end{pmatrix}$$

dependent

correlated

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{array}{cc} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{array}{c} HH \\ \\ TH \\ \end{array} \end{array}$$

Multiple Devices with Incomplete Information

$$A \otimes B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 B & a_2 B \\ a_3 B & a_4 B \end{pmatrix}$$

$$= \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_2 b_1 & a_2 b_2 \\ a_1 b_3 & a_1 b_4 & a_2 b_3 & a_2 b_4 \\ \multicolumn{2}{c}{a_3 B} & \multicolumn{2}{c}{a_4 B} \end{pmatrix}$$

There exist 4×4 transformations $C \neq A \otimes B$.

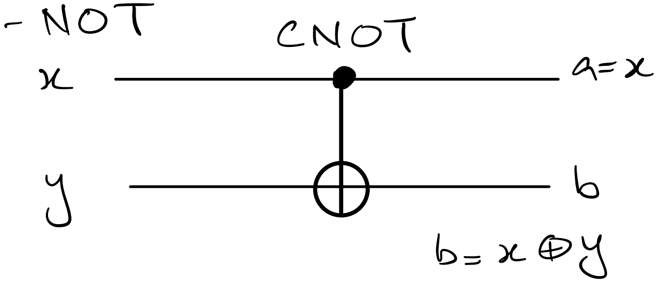
Multiple Devices with Incomplete Information

$$\overset{e_1}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \overset{e_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}, \overset{e_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}, \overset{e_4}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$$

Controlled-NOT

control

target



x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$$N := \begin{array}{cc} & \overbrace{\begin{array}{cccc} 00 & 01 & 10 & 11 \end{array}}^{xy} \\ ab\left\{\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}\right. & \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$$N \neq A \otimes B$$

How can a Biased Coin Simulate a Fair Coin?

Assume we have a coin $\begin{pmatrix} p \\ 1-p \end{pmatrix}$, $p \neq 0$, $p \neq \frac{1}{2}$.

How can we simulate a fair coin?