QWORLD

INTRODUCTION TO QUANTUM SYSTEMS

Jibran Rashid

Quantum Bit (Qubit)

Modelling a quantum system X with $\Sigma = \{0, 1\}$

Central Claim of Quantum Physics

To describe an isolated quantum system we need to give an amplitude ($\alpha \in \mathbb{C}$) for each possible state $\sigma \in \Sigma$.

Classical

$$\hat{v} = \underbrace{p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{convex combination}}$$

$p \geq 0$

Quantum

$$\hat{v} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\alpha, \beta \in \mathbb{C}$

What are the restrictions on α, β

Moving forward assume $\alpha, \beta \in \mathbb{R}$

Getting Probabilities from Amplitudes

Born Rule

Probability to observe a particular outcome,

e.g., Prob(X is in state 0) is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{matrix} \alpha^2 \longrightarrow Pr(X=0) \\ \beta^2 \longrightarrow Pr(X=1) \end{matrix}$$

assuming

$\alpha, \beta \in \mathbb{R}$

$$\hat{v} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha^2 + \beta^2 = 1$$

Quantum Bit (Qubit)

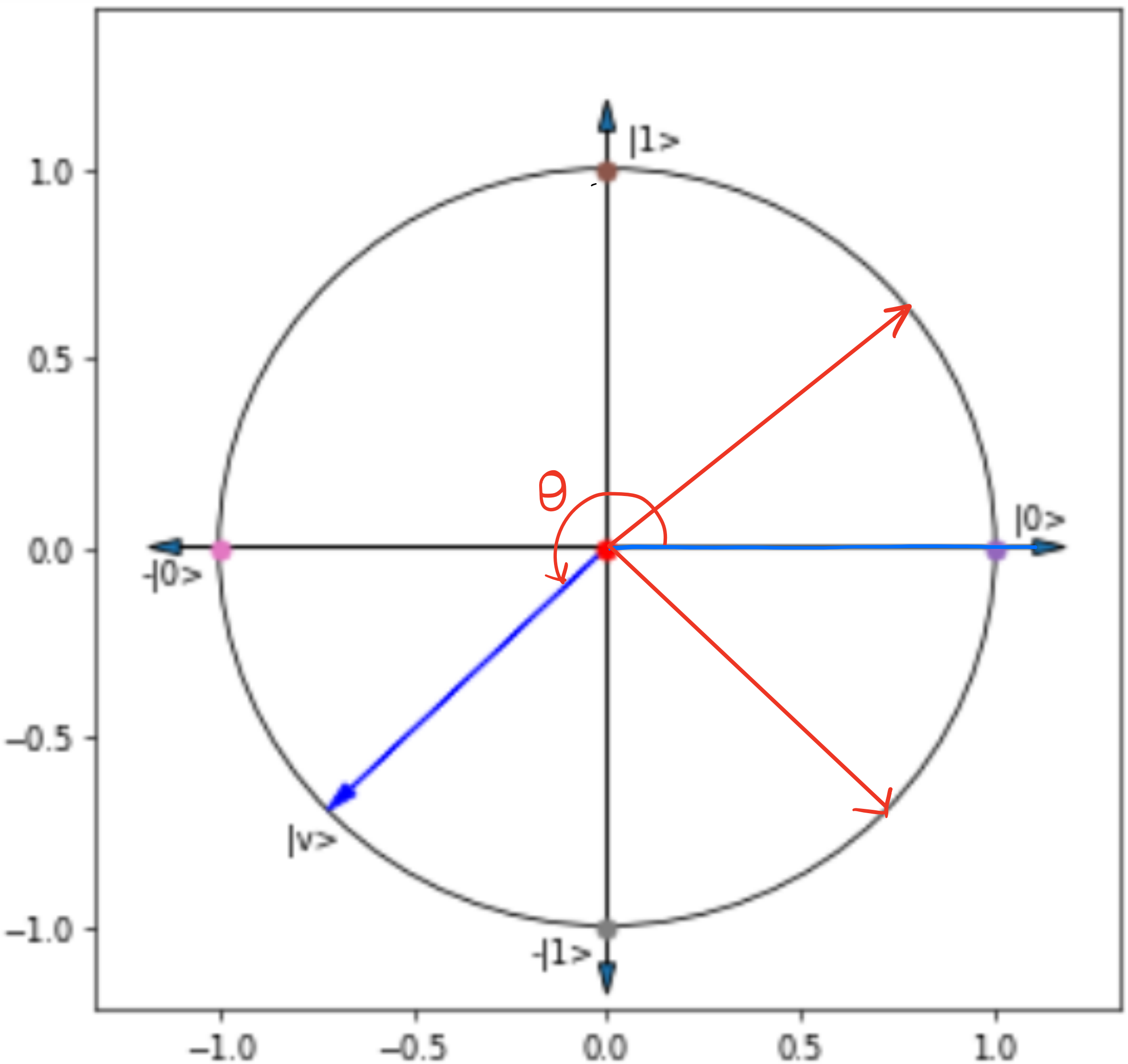
For real amplitudes, we can describe the state as

$$\hat{v} = \underbrace{\cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{superposition}}, \quad \theta \in [0, 2\pi)$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\hat{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Qubit on a Unit Circle



Quantum Operations

$U \hat{v} = \hat{w}$

$H^T = H$

Symmetric

$U \hat{v} = \hat{w}$

$$U \begin{pmatrix} \vdots \\ \alpha_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \beta_i \\ \vdots \end{pmatrix}$$

$$\sum_i \alpha_i^2 = 1 = \sum_i \beta_i^2$$

i) $U^T U = \mathbb{1}$ → identity, $U^T = U^{-1}$

ii) U^T is the inverse of U,

U is reversible

In general, Unitary transformations.

Quantum Operations

Ket Notation

$\underbrace{|\cdot\rangle}_{\text{ket}} \equiv$ column vector

$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{v}$

$|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underbrace{\langle\cdot|}_{\text{bra}} \equiv$ row vector

$\langle 0| \equiv \begin{pmatrix} 1 & 0 \end{pmatrix}$

$\langle 1| \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}$

$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

$\gamma|0\rangle + \delta|1\rangle = |\varphi\rangle$

$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$

$\underbrace{\langle\psi|\varphi\rangle}_{\hat{v}\cdot\hat{w}} \longrightarrow \langle\psi|\,|\varphi\rangle$

$(\alpha\langle 0| + \beta\langle 1|)(\gamma|0\rangle + \delta|1\rangle)$

$(\alpha\gamma\langle 0|0\rangle + \alpha\delta\langle 0|1\rangle + \beta\gamma\langle 1|0\rangle + \beta\delta\langle 1|1\rangle)$

$= \alpha\gamma + \beta\delta$

$\{|0\rangle, |1\rangle\} \longrightarrow$ Computational Basis

Standard

Quantum Operations

Hadamard Matrix

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H^T H = \mathbb{1}$

$H^T = H$

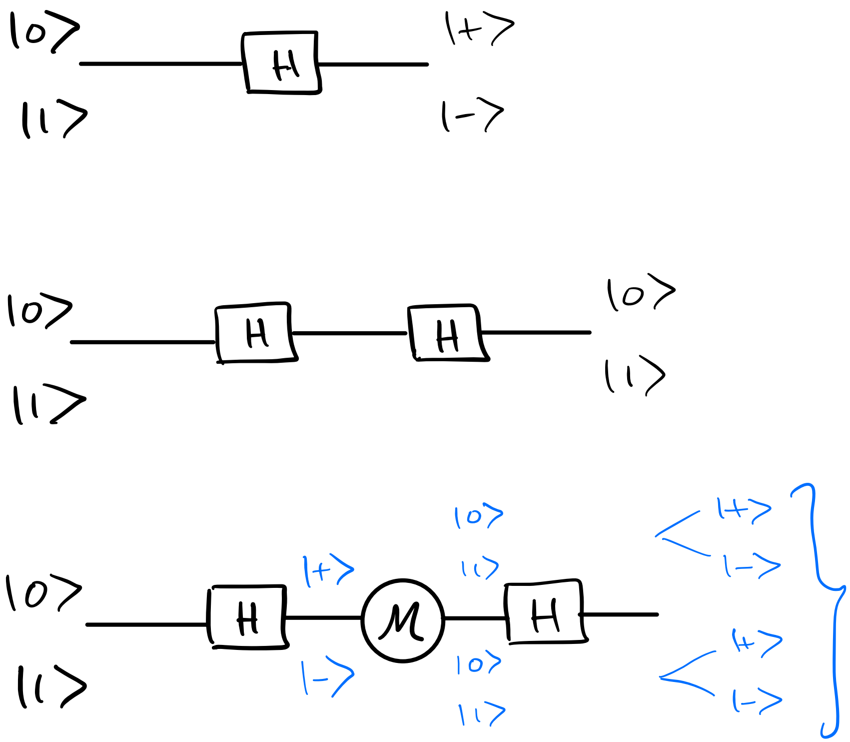
$F = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) = |+\rangle$ $\quad \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

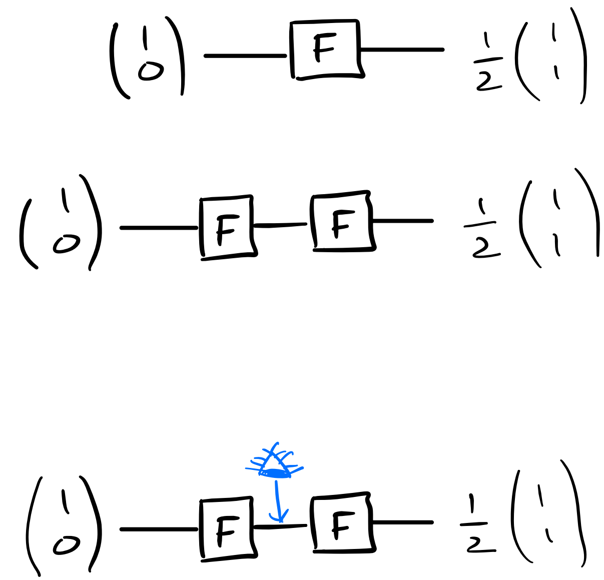
$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) = |-\rangle$

$\{ |+\rangle, |-\rangle \}$ Hadamard basis

Classical vs Quantum Coin Flipping



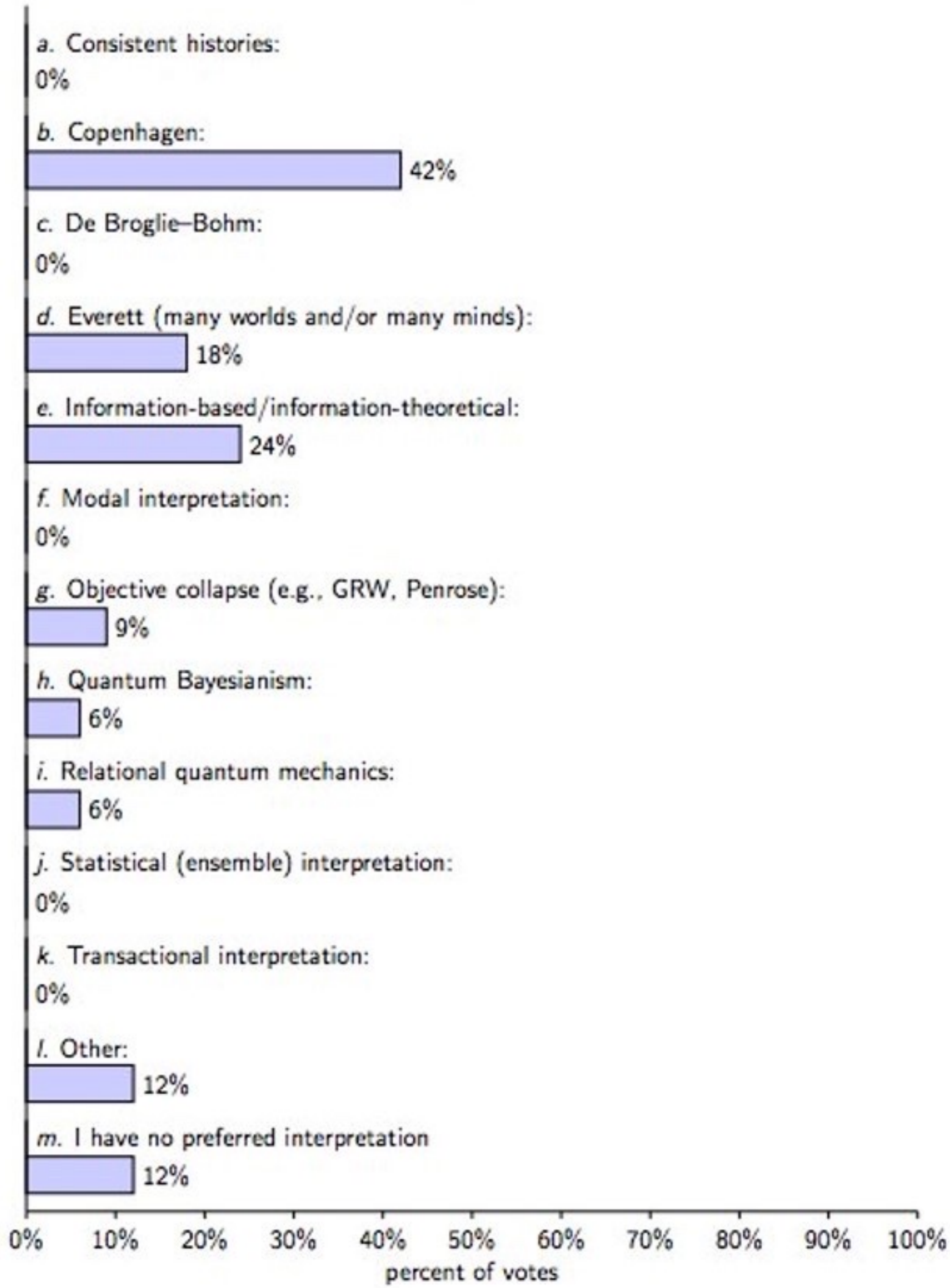
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$



$$F = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$F\begin{pmatrix}p\\ 1-p\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Question 12: What is your favorite interpretation of quantum mechanics?



Multiple Qubits

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\ , \quad |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$

$$|\psi\rangle \otimes |\varphi\rangle = |\psi\rangle|\varphi\rangle = |\psi\varphi\rangle = |\chi\rangle$$
$$= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$
$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \equiv \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

When does a quantum state admit an efficient simulation on a quantum computer?

Assume a system of n quantum bits.

The state at any given time should be of form

$$|\psi\rangle = \underbrace{(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) \otimes \cdots \otimes (\alpha_n|0\rangle + \beta_n|1\rangle)}_{\text{Seperable}}$$