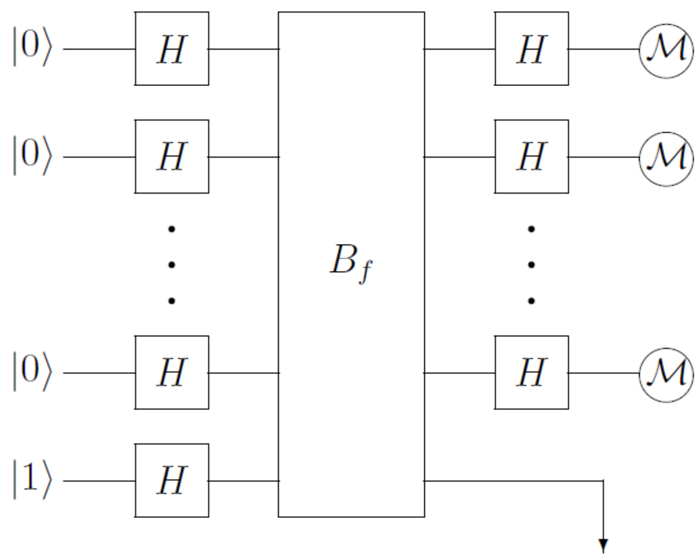
Deutsch Jozsa Algorithm

1. f is **constant**. In other words, either $f(x) = 0$ for all $x \in \{0,1\}^n$ or $f(x) = 1$ for all $x \in \{0,1\}^n$.
2. f is **balanced**. This means that the number of inputs $x \in \{0,1\}^n$ for which the function takes values 0 and 1 are the same:

$$|\{x \in \{0,1\}^n : f(x) = 0\}| = |\{x \in \{0,1\}^n : f(x) = 1\}| = 2^{n-1}.$$

$2^{n-1} + 1$



Classical # of queries for zero error = $2^{n-1} + 1$

$x_1 x_2 \dots x_n$	$f(\ )$
0 0 .. 0	
⋮	
1 1 ... 1	

2^n

Hadamard on n Qubits

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \quad H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$H|a\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a}|1\rangle\right) \quad a \in \{0,1\}$$

$a, b \in \{0,1\}$

$$= \frac{1}{\sqrt{2}} \sum_{b\in\{0,1\}} (-1)^{a\cdot b}|b\rangle$$

$a_1, a_2 \in \{0,1\}$

$b_1, b_2 \in \{0,1\}$

$$H \otimes H \underbrace{|a_1 a_2\rangle}_{a} = \left(\frac{1}{\sqrt{2}} \sum_{b_1} (-1)^{a_1 \cdot b_1} |b_1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{b_2} (-1)^{a_2 \cdot b_2} |b_2\rangle\right)$$

$$= \frac{1}{2} \sum_{b\in\{0,1\}^2} (-1)^{a\cdot b} |b\rangle$$

$a \cdot b = a_1 b_1 \oplus a_2 b_2$

$a, b \in \{00, 01, 10, 11\}$

$$H^{\otimes n}|a\rangle = \frac{1}{\sqrt{2^n}} \sum_{\hat{b}} (-1)^{\hat{a}\cdot\hat{b}} |\hat{b}\rangle$$

$\hat{a}, \hat{b} \in \{0,1\}^n$

Deutsch Jozsa Algorithm

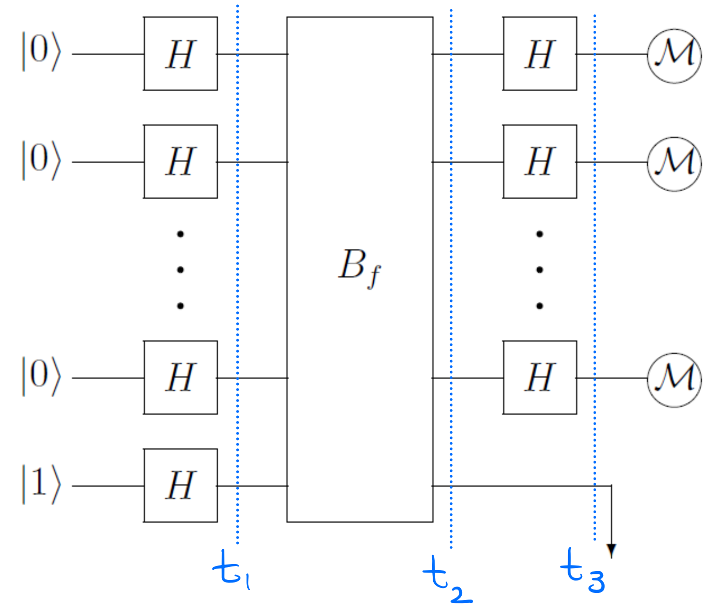
$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle|-\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} |x\rangle|-\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} \left(H^{\otimes n}|x\rangle\right) |-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y} |y\rangle\right) |-\rangle$$

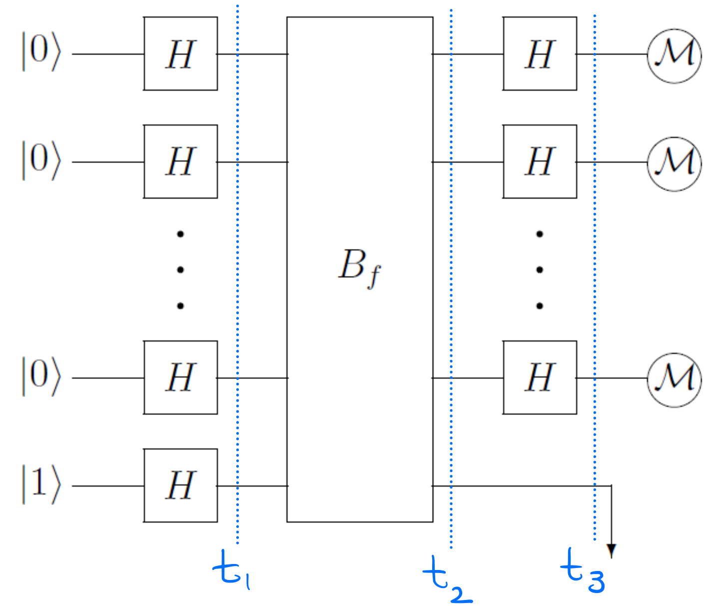
$$= \left[\sum_{y} \left(\frac{1}{2^n} \sum_{x} (-1)^{f(x)\oplus x\cdot y}\right) |y\rangle\right] |-\rangle$$



$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x_1y_1+\cdots+x_ny_n} |y\rangle$$

Deutsch Jozsa Algorithm

$$\left[\sum_y \left(\frac{1}{2^n}\sum_x (-1)^{f(x)\oplus x\cdot y}\right)|y\rangle\right]|-\rangle$$



$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y\in\{0,1\}^n}(-1)^{x_1y_1+\cdots+x_ny_n}|y\rangle$$

Let us figure out probability for measuring

$$y = 0^n = \underbrace{00\ldots0}_{n \text{ times}}$$

$$\left(\frac{1}{2^n}\sum_x (-1)^{f(x)\oplus x\cdot y}\right) \longrightarrow \text{amplitude for a given } |y\rangle$$

For $y = 0^n$ the probability for $|0^n\rangle$ is given by

$$\left(\frac{1}{2^n}\sum_x (-1)^{f(x)}\right)^2 = \begin{cases} 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$

Bernstein-Vazirani Problem

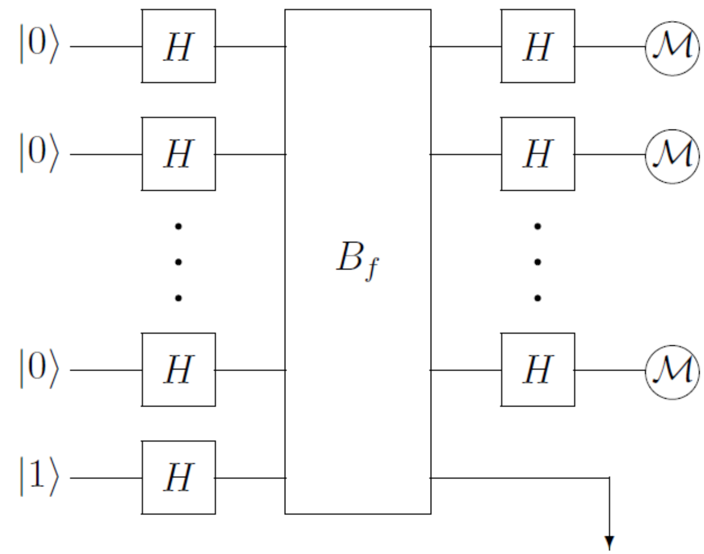
Suppose a function $f : \{0,1\}^n \rightarrow \{0,1\}$ is given as a black-box in the usual way, i.e., as a unitary transformation B_f that acts as follows for all $x \in \{0,1\}^n$ and $y \in \{0,1\}$:

$$B_f : |x\rangle\,|y\rangle \mapsto |x\rangle\,|y \oplus f(x)\rangle\,.$$

This time you are promised that there exists some string $s \in \{0,1\}^n$ such that $f(x) = s \cdot x$ for all $x \in \{0,1\}^n$, where

$s \in \{0,1\}^n$

$$s \cdot x = \sum_{i=1}^{n} s_i x_i \pmod 2.$$



$n = 3,\quad s.x = s_1 x_1 \oplus s_2 \cdot x_2 \ldots \oplus s_n x_n$

s_1	s_2	s_3	$f(x)$
0	0	0	0
0	0	1	x_3
0	1	0	x_2
0	1	1	$x_2 \oplus x_3$
⋮			⋮
1	1	1	$x_1 \oplus x_2 \oplus x_3$

Classical Query complexity

$O(n)$

x_1	x_2	x_3	$f(x)$
1	0	0	s_1
0	1	0	s_2
0	0	1	s_3

$\left.\begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix}\right\} s$

x_2	x_3	$x_2 \oplus x_3$
0	0	0
0	1	1
1	0	1
1	1	0

Bernstein-Vazirani Problem

Suppose a function $f : \{0,1\}^n \to \{0,1\}$ is given as a black-box in the usual way, i.e., as a unitary transformation B_f that acts as follows for all $x \in \{0,1\}^n$ and $y \in \{0,1\}$:

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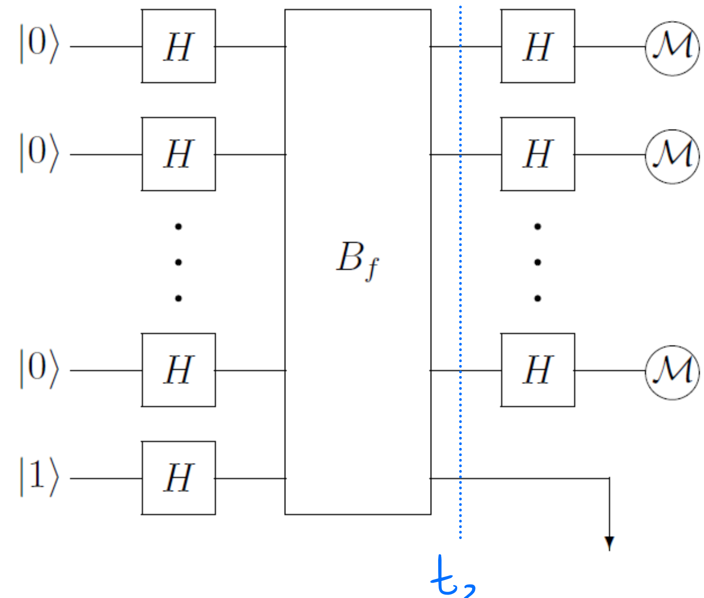

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle\, |-\rangle$$

$$H^{\otimes n}\, |\psi_2\rangle$$

$$|\psi_3\rangle = |s\rangle\, |-\rangle$$

$$H^{\otimes n}\, |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n} |y\rangle$$

$$H^{\otimes n}\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{s \cdot x} |x\rangle \right) = |s\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Finding Patterns

① Make superposition of all inputs

$$H^{\otimes n} |0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Finding Patterns

1. Make superposition of all inputs

$$H^{\otimes n} |0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

2. Get answers in the amplitude

B_f gives $\frac{1}{\sqrt{N}} \sum_x (-1)^{f(x)} |x\rangle$

Call $F(x) = (-1)^{f(x)}$

$F : \{0,1\}^n \longrightarrow \{\pm 1\}$

$0 \longrightarrow 1$

$1 \longrightarrow -1$

Loading up data in the vector

$$\frac{1}{\sqrt{N}} \begin{bmatrix} F(00..0) \\ F(00...1) \\ \vdots \\ F(11...1) \end{bmatrix}$$

Finding Patterns

1. Make superposition of all inputs
2. Get answers in the amplitude
3. Create interference

$$H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{N}}\sum_{x\in\{0,1\}^n}|x\rangle$$

B_f gives $\frac{1}{\sqrt{N}}\sum_x (-1)^{f(x)}|x\rangle$

Call $F(x) = (-1)^{f(x)}$

$F: \{0,1\}^n \longrightarrow \{\pm 1\}$

$0 \longrightarrow 1$

$1 \longrightarrow -1$

$H^{\otimes n}$ again

$$H^{\otimes n}\left(\frac{1}{\sqrt{N}}\sum_x F(x)|x\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\sum_x F(x)\, H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}}\sum_s \,?\, |s\rangle$$

Loading up data in the vector

$$\frac{1}{\sqrt{N}}\begin{bmatrix} F(00\ldots0) \\ F(00\ldots1) \\ \vdots \\ F(11\ldots1) \end{bmatrix}$$